CHAPTER 15
COMPLEX NUMBERS

In certain calculations in mathematics and related sciences, it is necessary to perform operations with numbers unlike any mentioned thus far in this course. These numbers, unfortunately called "imaginary" numbers by early mathematicians, are quite useful and have a very real meaning in the physical sense. The number system, which consists of ordinary numbers and imaginary numbers, is called the COMPLEX NUMBER system. Complex numbers are composed of a "real" part and an "imaginary" part.

This chapter is designed to explain imaginary numbers and to show how they can be combined with the numbers we already know.

REAL NUMBERS

The concept of number, as has been noted in previous chapters, has developed gradually. At one time the idea of number was limited to positive whole numbers.

The concept was broadened to include positive fractions; numbers that lie between the whole numbers. At first, fractions included only those numbers which could be expressed with terms that were integers. Since any fraction may be considered as a ratio, this gave rise to the term RATIONAL NUMBER, which is defined as any number which can be expressed as the ratio of two integers. (Remember that any whole number is an integer.)

It soon became apparent that these numbers were not enough to complete the positive number range. The ratio, π, of the circumference of a circle to its diameter, did not fit the concept of number thus far advanced, nor did such numbers as $\sqrt{2}$ and $\sqrt{3}$. Although decimal values are often assigned to these numbers, they are only approximations. That is, π is not exactly equal to 22/7 or to 3.142. Such numbers are called IRRATIONAL to distinguish them from the other numbers of the system. With rational and irrational numbers, the positive number system includes all the numbers from zero to infinity in a positive direction.

Since the number system was not complete with only positive numbers, the system was expanded to include negative numbers. The idea of negative rational and irrational numbers to minus infinity was an easy extension of the system.

Rational and irrational numbers, positive and negative to $\pm$ infinity as they have been presented in this course, comprise the REAL NUMBER system. The real number system is pictured in figure 15-1.

OPERATORS

As shown in a previous chapter, the plus sign in an expression such as 5 + 3 can stand for either of two separate things: It indicates the positive number 3, or it indicates that +3 is to be added to 5; that is, it indicates the operation to be performed on +3.

Likewise, in the problem 5 - 3, the minus sign may indicate the negative number -3, in which case the operation would be addition; that is, 5 + (-3). On the other hand, it may indicate the sign of operation, in which case +3 is to be subtracted from 5; that is, 5 - (+3).

Thus, plus and minus signs may indicate positive and negative numbers, or they may indicate operations to be performed.

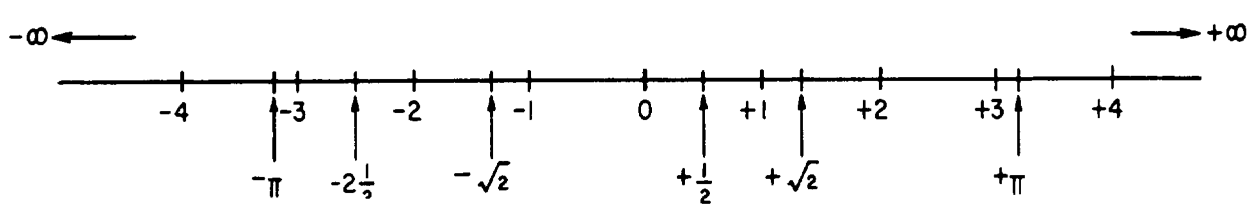

Figure 15-1.—The real number system.

IMAGINARY NUMBERS

The number line pictured in figure 15-1 represents all positive and negative numbers from plus infinity to minus infinity. However, there is a type of number which does not fit into the picture. Such a number occurs when we try to solve the following equation:

$$x^2 + 4 = 0$$
$$x^2 = -4$$
$$x = \pm \sqrt{-4}$$

Notice the distinction between this use of the radical sign and the manner in which it was used in chapter 7. Here, the ± symbol is included with the radical sign to emphasize the fact that two values of x exist. Although both roots exist, only the positive one is usually given. This is in accordance with usual mathematical convention.

The equation

$$x = \pm \sqrt{-4}$$

raises an interesting question:

What number multiplied by itself yields -4? The square of -2 is +4. Likewise, the square of +2 is +4. There is no number in the system of real numbers that is the square root of a negative number. The square root of a negative number came to be called an IMAGINARY NUMBER. When this name was assigned the square roots of negative numbers, it was natural to refer to the other known numbers as the REAL numbers.

IMAGINARY UNIT

To reduce the problem of imaginary numbers to its simplest terms, we proceed as far as possible using ordinary numbers in the solution. Thus, we may write $\sqrt{-4}$ as a product

$$\sqrt{-1 \cdot 4} = \sqrt{4}\sqrt{-1}$$
$$= \pm 2 \sqrt{-1}$$

Likewise,

$$\sqrt{-5} = \sqrt{5}\sqrt{-1}$$

Also,

$$3\sqrt{-7} = 3\sqrt{7}\sqrt{-1}$$

Thus, the problem of giving meaning to the square root of any negative number reduces to that of finding a meaning for $\sqrt{-1}$.

The square root of minus 1 is designated i by mathematicians. When it appears with a coefficient, the symbol i is written last unless the coefficient is in radical form. This convention is illustrated in the following examples:

$$\pm 2\sqrt{-1} = \pm 2i$$
$$\sqrt{5}\sqrt{-1} = i\sqrt{5}$$
$$3\sqrt{7}\sqrt{-1} = 3i\sqrt{7}$$

The symbol i stands for the imaginary unit $\sqrt{-1}$. An imaginary number is any real multiple, positive or negative, of i. For example, -7i, +7i, i $\sqrt{15}$, and bi are all imaginary numbers.

In electrical formulas the letter i denotes current. To avoid confusion, electronic technicians use the letter j to indicate $\sqrt{-1}$ and call it "operator j." The name "imaginary" should be thought of as a technical mathematical term of convenience. Such numbers have a very real purpose in the physical sense. Also it can be shown that ordinary mathematical operations such as addition, multiplication, and so forth, may be performed in exactly the same way as for the so-called real numbers.

Practice problems. Express each of the following as some real number times i:

1. $\sqrt{-16}$	3. $\sqrt{-5}$	5. $\sqrt{-25}$
2. $2\sqrt{-1}$	4. $\frac{d}{f}\sqrt{-f^2}$	6. $\sqrt{-\frac{9}{16}}$

Answers:

1. 4i	3. $i\sqrt{5}$	5. 5i
2. 2i	4. di	6. $\frac{3}{4}i$

Powers of the Imaginary Unit

The following examples illustrate the results of raising the imaginary unit to various powers:

$$i = \sqrt{-1}$$
$$i^2 = \sqrt{-1}\sqrt{-1}, \text{ or } -1$$
$$i^3 = i^2 i = -1i, \text{ or } -i$$
$$i^4 = i^2 i^2 = -1 \cdot -1 = +1$$
$$i^{-1} = \frac{1}{i} = \frac{i}{i^2} = \frac{i}{-1} = -i$$

We see from these examples that an even power of i is a real number equal to +1 or -1. Every odd power of i is imaginary and equal to i or -i. Thus, all powers of i reduce to one of the following four quantities: $\sqrt{-1}$, -1, $-\sqrt{-1}$, or +1.

GRAPHICAL REPRESENTATION

Figure 15-1 shows the real numbers represented along a straight line, the positive numbers extending from zero to the right for an infinite distance, and the negative numbers extending to the left of zero for an infinite distance. Every point on this line corresponds to a real number, and there are no gaps between them. It follows that there is no possibility of representing imaginary numbers on this line.

Earlier, we noted that certain signs could be used as operators. The plus sign could stand for the operation of addition. The minus sign could stand for the operation of subtraction. Likewise, it is easy to explain the imaginary number i graphically as an operator indicating a certain operation is to be performed on the number of which it is the coefficient.

If we graphically represent the length, n, on the number line pictured in figure 15-2 (A), we start at the point 0 and measure to the right (positive direction) a distance representing n units. If we multiply n by -1, we may represent the result -n by measuring from 0 in a negative direction a distance equal to n units.

Graphically, multiplying a real number by -1 is equivalent to rotating the line that represents the number about the point 0 through 180° so that the new position of n is in the opposite direction and a distance n units from 0. In this case we may think of -1 as the operator that rotates n through two right angles to its new position (fig. 15-2 (B)).

As we have shown, $i^2 = -1$. Therefore, we have really multiplied n by i^2, or i x i. In other words, multiplying by -1 is the same as multiplying by i twice in succession. Logically, if we multiplied n by i just once, the line n would be rotated only half as much as before—that is, through only one right angle, or 90°. The new segment ni would be measured in a direction 90° from the line n. Thus, i is an operator that rotates a number through one right angle. (See fig. 15-3.)

We have shown previously that a positive number may have two real square roots, one positive and one negative. For example, $\sqrt{9} = \pm 3$.

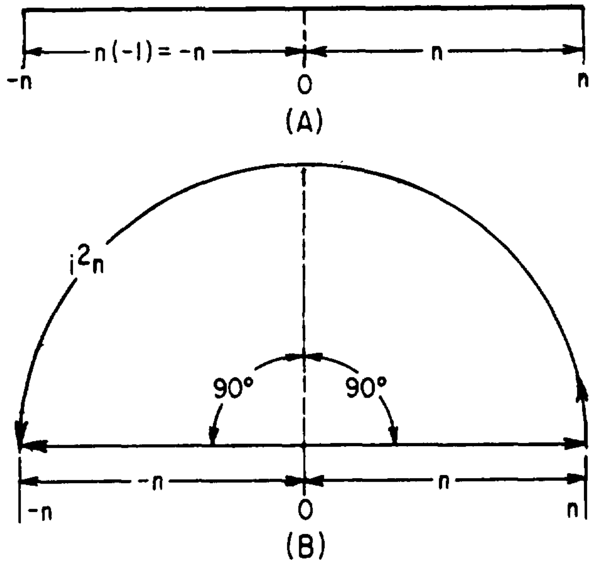

Figure 15-2.—Graphical multiplication by -1 and by operator i^2.

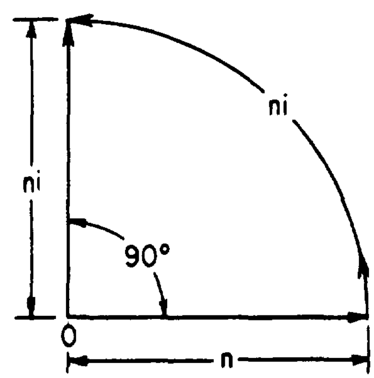

Figure 15-3.—Graphical multiplication by operator i.

We also saw that an imaginary number may have two roots. For example, $\sqrt{-4}$ is equal to $\pm 2i$. When the operator -1 graphically rotates a number, it may do so in a counterclockwise or a clockwise direction. Likewise, the operator i may graphically rotate a number in either direction. This fact gives meaning to numbers such as $\pm 2i$. It has been agreed that a number multiplied by +i is to be rotated 90° in a counterclockwise direction. A number multiplied by -i is to be rotated 90° in a clockwise direction.

In figure 15-4, +2i is represented by rotating the line that represents the positive real number 2 through 90° in a counterclockwise direction. It follows that -2i is represented by rotating the line that represents the positive real number 2 through 90° in a clockwise direction.

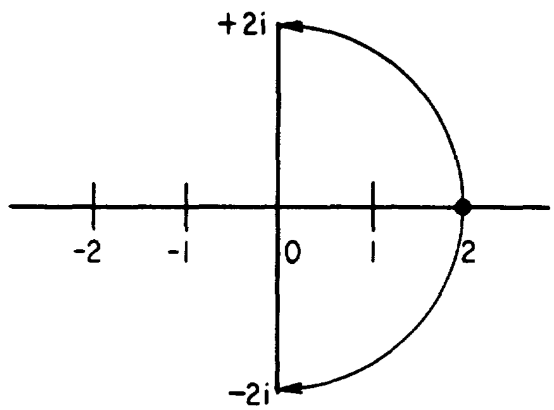

Figure 15-4.—Graphical representation of ±2i.

In figure 15-5, notice that the idea of i as an operator agrees with the concept advanced concerning the powers of i. Thus, i rotates a number through 90°; i^2 or -1 rotates the number through 180°, and the number is real and negative; i^3 rotates the number through 270°, which has the same effect as -i; and i^4 rotates the number through 360°, and the number is once again positive and real.

THE COMPLEX PLANE

All imaginary numbers may be represented graphically along a line extending through zero and perpendicular to the line representing the real numbers. This line may be considered infinite in both the positive and negative directions, and all multiples of i may be represented on it. This graph is similar to the rectangular coordinate system studied earlier.

In this system, the vertical or y axis is called the axis of imaginaries, and the horizontal or x axis is called the axis of reals. In the rectangular coordinate system, real numbers are laid off on both the x and y axes and the plane on which the axes lie is called the real plane. When the y axis is the axis of imaginaries, the plane determined by the x and y axes is called the COMPLEX PLANE (fig. 15-6).

In any system of numbers a unit is necessary for counting. Along the real axis, the unit is the number 1. As shown in figure 15-6, along the imaginary axis the unit is i. Numbers that lie along the imaginary axis are called PURE IMAGINARIES. They will always be some multiple of i, the imaginary unit. The numbers 5i, $3i\sqrt{2}$, and $\sqrt{-7}$ are examples of pure imaginaries.

NUMBERS IN THE COMPLEX PLANE

All numbers in the complex plane are complex numbers, including reals and pure imaginaries. However, since the reals and imaginaries have

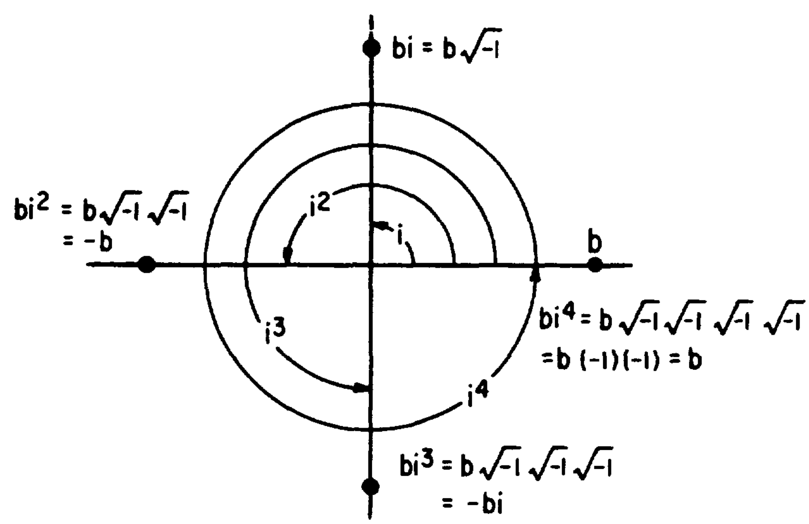

Figure 15-5.—Operation with powers of i.

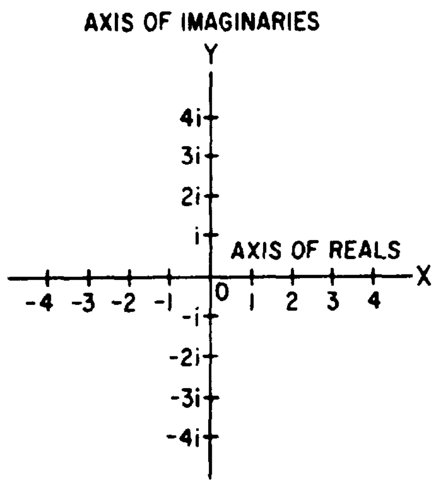

Figure 15-6.—The complex plane.

the special property of being located on the axes, they are usually identified by their distinguishing names.

The term complex number has been defined as the indicated sum or difference of a real number and an imaginary number.

For example, $3 + 5\sqrt{-1}$ or $3 + 5i$, $2 - 6i$, and $-2 + \sqrt{-5}$ are complex numbers. In the complex number $7 - i\sqrt{2}$, 7 is the real part and $-i\sqrt{2}$ is the imaginary part.

All complex numbers correspond to the general form $a + bi$, where a and b are real numbers. When a has the value 0, the real term disappears and the complex number becomes a pure imaginary. When b has the value of 0, the imaginary term disappears and the complex number becomes a real number. Thus, 4 may be thought of as $4 + 0i$, and $3i$ may be considered $0 + 3i$. From this we may reason that the real number and the pure imaginary number are special cases of the complex number. Consequently, the complex number may be thought of as the most general form of a number and can be construed to include all the numbers of algebra as shown in the chart in figure 15-7.

Plotting Complex Numbers

Complex numbers may easily be plotted in the complex plane. Pure imaginaries are plotted along the vertical axis, the axis of imaginaries, and real numbers are plotted along the horizontal axis, the axis of reals. It follows that other points in the complex plane must represent numbers that are part real and part imaginary; in other words, complex numbers. If we wish to plot the point $3 + 2i$, we note that the number is made up of the real number 3 and the imaginary number $2i$. Thus, as in figure 15-8, we measure along the real axis in a

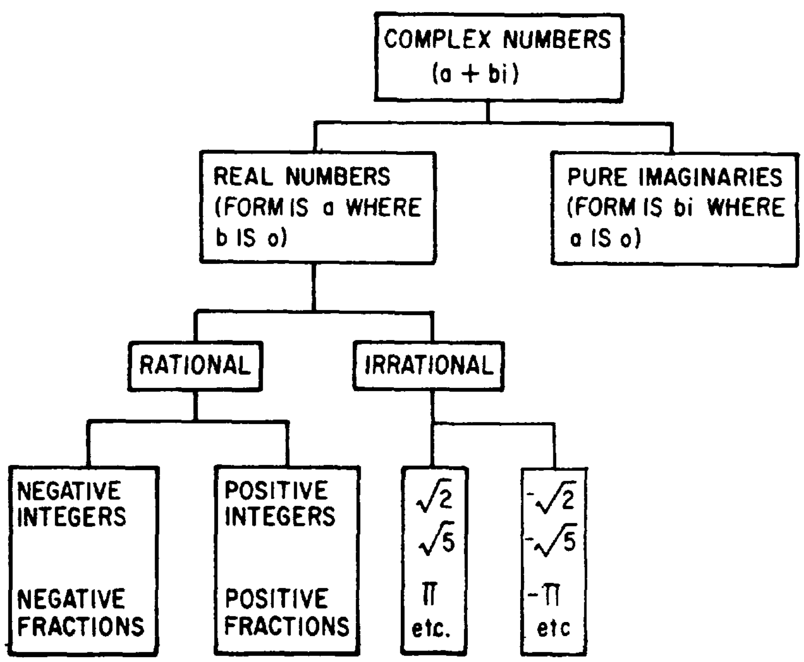

Figure 15-7.—The complex number system.

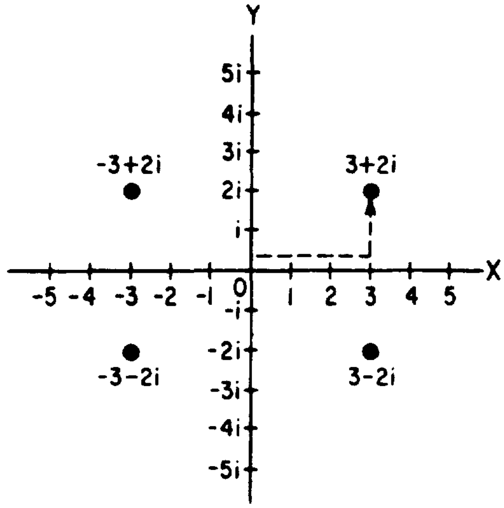

Figure 15-8.—Plotting complex numbers.

positive direction. At point (3, 0) on the real axis we turn through one right angle and measure 2 units up and parallel to the imaginary axis. Likewise, the number -3 + 2i is 3 units to the left and up 2 units; the number 3 - 2i is 3 units to the right and down 2 units; and the number -3 -2i is 3 units to the left and down 2 units.

Complex Numbers as Vectors

A vector is a directed line segment. A complex number represents a vector expressed in the RECTANGULAR FORM. For example, the complex number 6 + 8i in figure 15-9 may be considered as representing either the point P or the line OP. The real parts of the complex number (6 and 8) are the rectangular components of the vector. The real parts are the legs of the right triangle (sides adjacent to the right angle), and the vector OP is its hypotenuse (side opposite the right angle). If we merely wish to indicate the vector OP, we may do so by writing the complex number that represents it along the segment as in figure 15-9. This method not only fixes the position of point P, but also shows what part of the vector is imaginary (PA) and what part is real (OA).

If we wish to indicate a number that shows the actual length of the vector OP, it is necessary to solve the right triangle OAP for its hypotenuse. This may be accomplished by taking the square root of the sum of the squares of

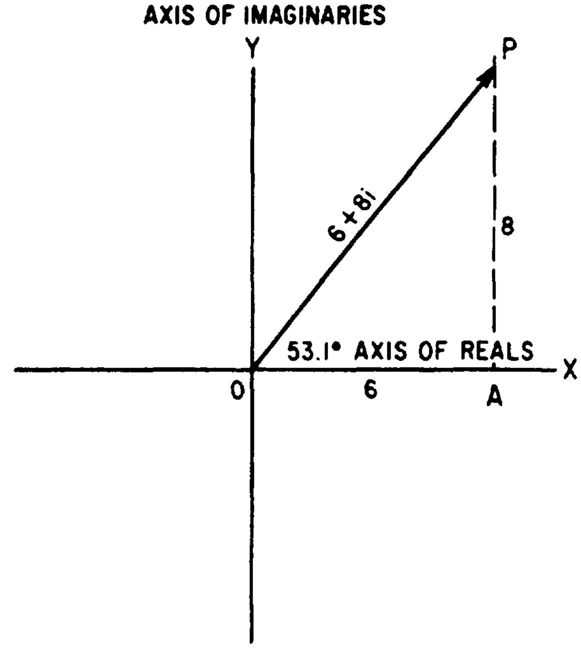

Figure 15-9.—A complex number shown as a vector.

the legs of the triangle, which in this case are the real numbers, 6 and 8. thus,

$$\begin{aligned} OP &= \sqrt{6^2 + 8^2} \\ &= \sqrt{100} \\ &= 10 \end{aligned}$$

However, since a vector has direction as well as magnitude, we must also show the direction of the segment; otherwise the segment OP could radiate in any direction on the complex plane from point 0. The expression $10\underline{/53.1^\circ}$ indicates that the vector OP has been rotated counterclockwise from the initial position through an angle of 53.1°. (The initial position in a line extending from the origin to the right along OX.) This method of expressing the vector quantity is called the POLAR FORM. The number represents the magnitude of the quantity, and the angle represents the position of the vector with respect to the horizontal reference, OX. Positive angles. represent counterclockwise rotation of the vector, and negative angles represent clockwise rotation. The polar form is generally simpler for multiplication and division, but its use requires a knowledge of trigonometry.

ADDITION AND SUBTRACTION OF COMPLEX NUMBERS

Pure imaginaries are added and subtracted in the same way as any other algebraic quantities. The coefficients of similar terms are added or subtracted algebraically, as follows:

$$4i + 3i = 7i$$
$$4i - 3i = i$$
$$4i - (-3i) = 7i$$

Likewise, complex numbers in the rectangular form are combined like any other algebraic polynomials. Add or subtract the coefficients of similar terms algebraically. If parentheses enclose the numbers, first remove the parentheses. Next, place the real parts together and the imaginary parts together. Collect terms. As examples, consider the following:

1. $$\begin{aligned}(2 - 3i) + (5 + 4i) &= 2 - 3i + 5 + 4i \\ &= 2 + 5 - 3i + 4i \\ &= 7 + i\end{aligned}$$

2. $$\begin{aligned}(2 - j3) - (5 + j4) &= 2 - j3 - 5 - j4 \\ &= 2 - 5 - j3 - j4 \\ &= -3 - j7\end{aligned}$$

In example 2, notice that the convention for writing operator j (the electronics form of the imaginary unit) with numerical coefficients is to place j first.

If the complex numbers are placed one under the other, the results of addition and subtraction appear as follows:

ADDITION	SUBTRACTION
$3 + 4\sqrt{-1}$	$a + jb$
$\underline{2 - 7\sqrt{-1}}$	$\underline{[\leftarrow c + jd}$
$5 - 3\sqrt{-1}$	$(a - c) + j(b - d)$

Practice problems. Add or subtract as indicated, in the following problems:

1. $(3a + 4i) + (0 - 2i)$
2. $(3 + 2i) + (-3 + 3i)$
3. $(a + bi) + (c + di)$
4. $(1 + 2\sqrt{-1}) + (-2 - 2\sqrt{-1})$
5. $(-5 + 3i) - (4 - 2i)$
6. $(a + bi) - (-c + di)$

Answers:

1. $3a + 2i$
2. $5i$
3. $a + c + (b + d)i$
4. -1
5. $-9 + 5i$
6. $a + c + (b - d)i$

MULTIPLICATION OF COMPLEX NUMBERS

Generally, the rules for the multiplication of complex numbers and pure imaginaries are the same as for other algebraic quantities. However, there is one exception that should be noted: The rule for multiplying numbers under radical signs does not apply to TWO NEGATIVE numbers. When at least one of two radicands is positive, the radicands can be multiplied immediately, as in the following examples:

$$\sqrt{2}\sqrt{3} = \sqrt{6}$$
$$\sqrt{2}\sqrt{-3} = \sqrt{-6}$$

When both radicands are negative, however, as in $\sqrt{-2}\sqrt{-3}$, an inconsistent result is obtained if we multiply both numbers under the radical signs immediately. To get the correct result, express the imaginary numbers first in terms of i, as follows:

$$\begin{aligned}\sqrt{-2}\sqrt{-3} &= i\sqrt{2} \cdot i\sqrt{3} \\ &= i^2\sqrt{2}\sqrt{3} \\ &= i^2\sqrt{6} \\ &= (-1)\sqrt{6} = -\sqrt{6}\end{aligned}$$

Multiplying complex numbers is equivalent to multiplying binomials in the manner explained previously. After the multiplication is performed, simplify the powers of i as in the following examples:

1. $$\begin{array}{l} 4 - i \\ \underline{3 + i} \\ 12 - 3i \\ \underline{\quad + 4i - i^2} \\ 12 + i - i^2 = 12 + i - (-1) \\ \qquad\qquad\quad = 13 + i \end{array}$$

2. $(-6 + 5\sqrt{-7})\,(8 - 2\sqrt{-7})$

$$= (-6 + 5i\sqrt{7})\,(8 - 2i\sqrt{7})$$
$$= -48 + 40i\sqrt{7} + 12i\sqrt{7} - 10(7)i^2$$
$$= -48 + 52i\sqrt{7} + 70$$
$$= 22 + 52i\sqrt{7}$$

Practice problems. Perform the indicated operations:

1. $\sqrt{-9}\,\sqrt{-16}$
2. $\sqrt{-2}\,\sqrt{18}$
3. $\sqrt{-9}\,\sqrt{-4}$
4. $a\sqrt{-ba} \cdot \sqrt{-b}$
5. $(2 + 5i)\,(3 - 2i)$
6. $(a + \sqrt{-b})\,(a - \sqrt{-b})$
7. $(-2 + \sqrt{-4})\,(-1 + \sqrt{-4})$
8. $(8 - \sqrt{-7})\,(6 + \sqrt{-7})$

Answers:

1. -12	5. $16 + 11i$
2. $6i$	6. $a^2 + b$
3. -6	7. $-2 - 6i$
4. $-ab\sqrt{a}$	8. $55 + 2i\sqrt{7}$

CONJUGATES AND SPECIAL PRODUCTS

Two complex numbers that are alike except for the sign of their imaginary parts are called CONJUGATE COMPLEX NUMBERS. For example, $3 + 5i$ and $3 - 5i$ are conjugates. Either number is the conjugate of the other.

If one complex number is known, the conjugate can be obtained immediately by changing the sign of the imaginary part. The conjugate of $-8 + \sqrt{-10}$ is $-8 - \sqrt{-10}$. The conjugate of $-\sqrt{-6}$ is $\sqrt{-6}$.

The sum of two conjugate complex numbers is a real number, as illustrated by the following:

1. $(3 + j5) + (3 - j5) = 2(3) = 6$

2. $\left(-\frac{1}{2} + \frac{\sqrt{-3}}{2}\right) + \left(-\frac{1}{2} - \frac{\sqrt{-3}}{2}\right)$

$$= -\frac{1}{2} + \frac{\sqrt{3}}{2}i - \frac{1}{2} - \frac{\sqrt{3}}{2}i$$
$$= -\frac{1}{2} + \left(-\frac{1}{2}\right)$$
$$= -1$$

Product of Two Conjugates

The product of two conjugate complex numbers is a real number. Multiplying two conjugates is equivalent to finding the product of the sum and difference of two numbers.

Consider the following examples:

1. $(3 + j5)\,(3 - j5) = 3^2 - (j5)^2$

$$= 9 - 25(-1)$$
$$= 9 + 25$$
$$= 34$$

2. $\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = \left(-\frac{1}{2}\right)^2 - \left(\frac{\sqrt{3}}{2}i\right)^2$

$$= \frac{1}{4} - \left[\frac{3}{4}(-1)\right]$$
$$= \frac{1}{4} + \frac{3}{4}$$
$$= 1$$

Squaring a Complex Number

Squaring a complex number is equivalent to raising a binomial to the second power. For example:

$$(-6 - \sqrt{-25})^2 = (-6 - j5)^2$$
$$= [(-1) \cdot (6 + j5)]^2$$
$$= (-1)^2 \cdot (6^2 + j60 + j^2 25)$$
$$= 36 + j60 - 25$$
$$= 11 + j60$$

DIVISION OF COMPLEX NUMBERS

When dividing by a pure imaginary, the denominator may be rationalized and the problem thus simplified by multiplying both numerator and denominator by the denominator. Thus,

$$\frac{12}{\sqrt{-2}} = \frac{12}{i\sqrt{2}} \cdot \frac{i\sqrt{2}}{i\sqrt{2}}$$
$$= \frac{12i\sqrt{2}}{2i^2}$$
$$= \frac{6i\sqrt{2}}{-1}$$
$$= -6i\sqrt{2}$$

Division of complex numbers can be accomplished by multiplying the numerator and denominator by the number that is the conjugate of the denominator. This process is similar to the process of rationalizing a denominator in the case of real numbers that are irrational.

As an example, consider

$$\frac{5 - 2i}{3 + i}$$

The denominator is 3 + i. Its conjugate is 3 - i. Multiplying numerator and denominator by 3 - i gives

$$\frac{5 - 2i}{3 + i} \cdot \frac{3 - i}{3 - i} = \frac{15 - 11i + 2i^2}{9 - i^2}$$

$$= \frac{15 - 11i - 2}{9 + 1}$$

$$= \frac{13 - 11i}{10}$$

$$= \frac{13}{10} - \frac{11}{10}i$$

Practice problems. Rationalize the denominators and simplify:

1. $\frac{2\sqrt{-1}}{4 + 2\sqrt{-1}}$
2. $\frac{-2 + 4i}{-1 + 4i}$
3. $\frac{3 + \sqrt{-2}}{3 - \sqrt{-2}}$
4. $\frac{3}{1 - i\sqrt{3}}$
5. $\frac{1 - i}{2 - i}$
6. $\frac{8}{2 + \sqrt{-2}}$

Answers:

1. $\frac{2i + 1}{5}$
2. $\frac{18 + 4i}{17}$
3. $\frac{7 + 6i\sqrt{2}}{11}$
4. $\frac{3}{4} + \frac{3}{4}i\sqrt{3}$
5. $\frac{3 - i}{5}$
6. $\frac{8 - 4i\sqrt{2}}{3}$